THE K-RANK NUMERICAL RADII

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ABSTRACT. The k-rank numerical range $\Lambda_k(A)$ is expressed via an intersection of any countable family of numerical ranges $\{F(M_\nu^* A M_\nu)\}_{\nu\in\mathbb{N}}$ with respect to $n \times (n-k+1)$ isometries M_ν. This implication for $\Lambda_k(A)$ provides further elaboration of the k-rank numerical radii of A.

1. Introduction

Let $\mathcal{M}_n(\mathbb{C})$ be the algebra of $n \times n$ complex matrices and $k \geq 1$ be a positive integer. The *k-rank numerical range* $\Lambda_k(A)$ of a matrix $A \in \mathcal{M}_n$ is defined by

$$\begin{aligned} \Lambda_k(A) &= \{\lambda \in \mathbb{C} : X^* A X = \lambda I_k \text{ for some } X \in \mathcal{X}_k\} \\ &= \{\lambda \in \mathbb{C} : PAP = \lambda P \text{ for some } P \in \mathcal{Y}_k\}, \end{aligned}$$

where $\mathcal{X}_k = \{X \in \mathcal{M}_{n,k} : \ X^*X = I_k\}$ and $\mathcal{Y}_k = \{P \in \mathcal{M}_n : P = XX^*, X \in \mathcal{X}_k\}$. Note that $\Lambda_k(A)$ has been introduced as a versatile tool to solving a fundamental error correction problem in quantum computing [3, 4, 6, 7, 9].

For $k = 1$, $\Lambda_k(A)$ reduces to the classical *numerical range* of a matrix A,

$$\Lambda_1(A) \equiv F(A) = \{x^* A x : x \in \mathbb{C}^n, \, x^* x = 1\},$$

which is known to be a compact and convex subset of $\mathbb{C}$ [5], as well as the same properties hold for the set $\Lambda_k(A)$, for $k > 1$ [7, 9]. Associated with $\Lambda_k(A)$ are the *k-rank numerical radius* $r_k(A)$ and the *inner k-rank numerical radius* $\widetilde{r}_k(A)$, defined respectively, by

$$r_k(A) = \max\{|z| : z \in \partial\Lambda_k(A)\} \quad \text{and} \quad \widetilde{r}_k(A) = \min\{|z| : z \in \partial\Lambda_k(A)\}.$$

For $k = 1$, they yield the *numerical radius* and the *inner numerical radius*,

$$r(A) = \max\{|z| : z \in \partial F(A)\} \quad \text{and} \quad \widetilde{r}(A) = \min\{|z| : z \in \partial F(A)\},$$

respectively.

In the first section of this paper, $\Lambda_k(A)$ is proved to coincide with an indefinite intersection of numerical ranges of all the compressions of $A \in \mathcal{M}_n$ to $(n-k+1)$-dimensional subspaces, which has been also used in [3, 4]. Further elaboration led us to reformulate $\Lambda_k(A)$ in terms of an intersection of a countable family of numerical ranges. This result provides additional characterizations of $r_k(A)$ and $\widetilde{r}_k(A)$, which are presented in section 3.

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2. Alternative expressions of $\Lambda_k(A)$

Initially, the higher rank numerical range $\Lambda_k(A)$ is proved to be equal to an infinite intersection of numerical ranges.

Theorem 2.1. *Let $A \in \mathcal{M}_n(\mathbb{C})$. Then*

$$\Lambda_k(A) = \bigcap_{M \in \mathcal{X}_{n-k+1}} F(M^*AM) = \bigcap_{P \in \mathcal{Y}_{n-k+1}} F(PAP).$$

Proof. Denoting by $\lambda_1(H) \geq \ldots \geq \lambda_n(H)$ the decreasingly ordered eigenvalues of a hermitian matrix $H \in \mathcal{M}_n(\mathbb{C})$, we have [7]

$$\Lambda_k(A) = \bigcap_{\theta \in [0,2\pi)} e^{-\mathrm{i}\theta}\{z \in \mathbb{C} : \mathrm{Re} z \leq \lambda_k(H(e^{\mathrm{i}\theta}A))\}$$

where $H(\cdot)$ is the hermitian part of a matrix. Moreover, by Courant-Fisher theorem, we have

$$\lambda_k(H(e^{\mathrm{i}\theta}A)) = \min_{\dim \mathcal{S}=n-k+1} \max_{\substack{x \in \mathcal{S} \\ \|x\|=1}} x^* H(e^{\mathrm{i}\theta}A)x.$$

Denoting by $\mathcal{S} = span\{u_1, \ldots, u_{n-k+1}\}$, where $u_i \in \mathbb{C}^n$, $i = 1, \ldots, n-k+1$ are orthonormal vectors, then any unit vector $x \in \mathcal{S}$ is written in the form $x = My$, where $M = \begin{bmatrix} u_1 & \cdots & u_{n-k+1} \end{bmatrix} \in \mathcal{X}_{n-k+1}$ and $y \in \mathbb{C}^{n-k+1}$ is unit. Hence, we have

$$\begin{aligned}
\lambda_k(H(e^{\mathrm{i}\theta}A)) &= \min_M \max_{\substack{y \in \mathbb{C}^{n-k+1} \\ \|y\|=1}} y^*M^*H(e^{\mathrm{i}\theta}A)My \\
&= \min_M \max_{\substack{y \in \mathbb{C}^{n-k+1} \\ \|y\|=1}} y^*H(e^{\mathrm{i}\theta}M^*AM)y \\
&= \min_M \lambda_1(H(e^{\mathrm{i}\theta}M^*AM))
\end{aligned}$$

and consequently

$$\begin{aligned}
\Lambda_k(A) &= \bigcap_\theta e^{-\mathrm{i}\theta}\{z \in \mathbb{C} : \mathrm{Re} z \leq \min_M \lambda_1(H(e^{\mathrm{i}\theta}M^*AM))\} \\
&= \bigcap_M \bigcap_\theta e^{-\mathrm{i}\theta}\{z \in \mathbb{C} : \mathrm{Re} z \leq \lambda_1(H(e^{\mathrm{i}\theta}M^*AM))\} \\
&= \bigcap_{M \in \mathcal{X}_{n-k+1}} F(M^*AM).
\end{aligned}$$

Moreover, if we consider the $(n-k+1)$-rank orthogonal projection $P = MM^*$ of $\mathbb{C}^n$ onto the aforementioned space $\mathcal{S}$, then $x = Px$, for $x \in \mathcal{S}$ and $P\hat{x} = 0$, for $\hat{x} \notin \mathcal{S}$. Hence, we have

$$\Lambda_k(A) = \bigcap_{P \in \mathcal{Y}_{n-k+1}} F(PAP).$$

□

At this point, we should note that Theorem 2.1 provides a different and independent characterization of $\Lambda_k(A)$ than the one given in [6, Cor. 4.9]. We focus on the expression of $\Lambda_k(A)$ via the numerical ranges $F(M^*AM)$ (or $F(PAP)$), since it represents a more useful and advantageous procedure to determine and approximate the boundary of $\Lambda_k(A)$ numerically.

In addition, Theorem 2.1 verifies the "*convexity of* $\Lambda_k(A)$" through the convexity of the numerical ranges $F(M^*AM)$ (or $F(PAP)$), which is ensured by the Toeplitz-Hausdorff theorem. A different way of indicating that $\Lambda_k(A)$ is convex, is developed in [9]. For $k = n$, clearly $\Lambda_n(A) = \bigcap_{x\in\mathbb{C}^n,\|x\|=1} F(x^*Ax)$ and should be $\Lambda_n(A) \neq \emptyset$ *precisely* when A is scalar.

Motivated by the above, we present the main result of our paper, redescribing the higher rank numerical range as *a countable intersection of numerical ranges.*

Theorem 2.2. *Let $A \in \mathcal{M}_n$. Then for any countable family of orthogonal projections $\{P_\nu : \nu \in \mathbb{N}\} \subseteq \mathcal{Y}_{n-k+1}$ (or any family of isometries $\{M_\nu : \nu \in \mathbb{N}\} \subseteq \mathcal{X}_{n-k+1}$) we have*

$$\Lambda_k(A) = \bigcap_{\nu\in\mathbb{N}} F(P_\nu A P_\nu) = \bigcap_{\nu\in\mathbb{N}} F(M_\nu^* A M_\nu). \tag{2.1}$$

Proof. By Theorem 2.1, we have

$$[\Lambda_k(A)]^c = \mathbb{C} \setminus \Lambda_k(A) = \bigcup_{P\in\mathcal{Y}_{n-k+1}} [F(PAP)^c],$$

whereupon the family $\{F(PAP)^c : P \in \mathcal{Y}_{n-k+1}\}$ is an open cover of $[\Lambda_k(A)]^c$. Moreover, $[\Lambda_k(A)]^c$ is separable, as an open subset of the separable space $\mathbb{C}$ and then $[\Lambda_k(A)]^c$ has a countable base [8], which obviously depends on the matrix A. This fact guarantees that any open cover of $[\Lambda_k(A)]^c$ admits a countable subcover, leading to the relation

$$[\Lambda_k(A)]^c = \bigcup_{\nu\in\mathbb{N}} [F(P_\nu A P_\nu)^c],$$

i.e. leading to the first equality in (2.1). Taking into consideration that there exists a countable dense subset $\mathcal{J} \subseteq \mathcal{Y}_{n-k+1}$ with respect to the operator norm $\|\cdot\|$ and $P_\nu \in \mathcal{Y}_{n-k+1}$, for $\nu \in \mathbb{N}$, clearly, $\bigcap_{\nu\in\mathbb{N}} F(P_\nu A P_\nu) = \bigcap_{\nu\in\mathbb{N}, P_\nu\in\mathcal{J}} F(P_\nu A P_\nu)$. That is in (2.1), the family of orthogonal projections $\{P_\nu : \nu \in \mathbb{N}\}$ can be chosen independently of A. Moreover, due to $P_\nu = M_\nu M_\nu^*$, with $M_\nu \in \mathcal{X}_{n-k+1}$, we derive the second equality in (2.1). □

For a construction of a countable family of isometries $\{M_\nu : \nu \in \mathbb{N}\} \subseteq \mathcal{X}_{n-k+1}$, see also in the Appendix.

Furthermore, using the dual "max-min" expression of the k-th eigenvalue,

$$\lambda_k(H(e^{i\theta}A)) = \max_{\dim\mathcal{G}=k} \min_{\substack{x\in\mathcal{G}\\ \|x\|=1}} x^* H(e^{i\theta}A)x = \max_N \lambda_{\min}(H(e^{i\theta}N^*AN)),$$

where $N \in \mathcal{X}_k$, we have

$$\begin{aligned} \Lambda_k(A) &= \bigcap_{\theta} e^{-\mathrm{i}\theta}\{z \in \mathbb{C} : \mathrm{Re} z \le \max_N \lambda_k(H(e^{\mathrm{i}\theta}N^*AN))\} \\ &= \bigcup_N \bigcap_\theta e^{-\mathrm{i}\theta}\{z \in \mathbb{C} : \mathrm{Re} z \le \lambda_k(H(e^{\mathrm{i}\theta}N^*AN))\} \\ &= \bigcup_{N \in \mathcal{X}_k} \Lambda_k(N^*AN), \end{aligned} \tag{2.2}$$

and due to the convexity of $\Lambda_k(A)$, we establish

$$\Lambda_k(A) = \mathrm{co} \bigcup_{N \in \mathcal{X}_k} \Lambda_k(N^*AN), \tag{2.3}$$

where $\mathrm{co}(\cdot)$ denotes the convex hull of a set. Apparently, $\Lambda_k(N^*AN) \neq \emptyset$ if and only if $N^*AN = \lambda I_k$ [6] and then (2.3) is reduced to $\bigcup_N \Lambda_k(N^*AN) = \bigcup_N \{\lambda : N^*AN = \lambda I_k\} = \Lambda_k(A)$, where N runs all $n \times k$ isometries.

In spite of Theorem 2.2, $\Lambda_k(A)$ cannot be described as a countable union in (2.2), because if

$$\Lambda_k(A) = \bigcup_{\nu \in \mathbb{N}} \{\Lambda_k(N_\nu^* A N_\nu) : N_\nu \in \mathcal{X}_k\} = \bigcup_{\nu \in \mathbb{N}} \{\lambda_\nu : N_\nu^* A N_\nu = \lambda_\nu I_k,\ N_\nu \in \mathcal{X}_k\},$$

then $\Lambda_k(A)$ should be a countable set, which is not true.

3. Properties of $r_k(A)$ and $\widetilde{r}_k(A)$

In this section, we characterize the k-rank numerical radius $r_k(A)$ and the inner k-rank numerical radius $\widetilde{r}_k(A)$. Motivated by Theorem 2.2, we present the next two results.

Theorem 3.1. *Let $A \in \mathcal{M}_n$ and $\mathcal{J}_\nu(A) = \bigcap_{p=1}^{\nu} F(M_p^* A M_p)$, where $M_p \in \mathcal{X}_{n-k+1}$. Then*

$$r_k(A) = \lim_{\nu \to \infty} \sup\{|z| : z \in \mathcal{J}_\nu(A)\} = \inf_{\nu \in \mathbb{N}} \sup\{|z| : z \in \mathcal{J}_\nu(A)\}.$$

Proof. By Theorem 2.2, we have

$$\Lambda_k(A) = \bigcap_{\nu=1}^{\infty} \mathcal{J}_\nu(A) \subseteq \mathcal{J}_\nu(A) \subseteq F(A) \subseteq \mathcal{D}(0, \|A\|_2), \tag{3.1}$$

for all $\nu \in \mathbb{N}$, where the sequence $\{\mathcal{J}_\nu(A)\}_{\nu \in \mathbb{N}}$ is nonincreasing and $\mathcal{D}(0, \|A\|_2)$ is the circular disc centered at the origin with radius the spectral norm $\|A\|_2$ of $A \in \mathcal{M}_n$. Clearly,

$$r_k(A) = \max_{z \in \bigcap_{\nu=1}^{\infty} \mathcal{J}_\nu(A)} |z| \le \sup_{z \in \mathcal{J}_\nu(A)} |z| \le r(A) \le \|A\|_2,$$

then the nonincreasing and bounded sequence $q_\nu = \sup\{|z| : z \in \mathcal{J}_\nu(A)\}$ converges. Therefore

$$r_k(A) \le \lim_{\nu \to \infty} q_\nu = q_0.$$

We shall prove that the above inequality is actually an equality. Assume that $r_k(A) < q_0$. In this case, there is $\varepsilon > 0$, where $r_k(A) + \varepsilon < q_0 \le q_\nu$ for all

$\nu \in \mathbb{N}$. Then we may find a sequence $\{\zeta_\nu\} \subseteq \mathcal{J}_\nu(A)$ such that $q_0 \leq |\zeta_\nu|$ for all $\nu \in \mathbb{N}$. Due to the boundedness of the set $\mathcal{J}_\nu(A)$, the sequence $\{\zeta_\nu\}$ contains a subsequence $\{\zeta_{\rho_\nu}\}$ converging to $\zeta_0 \in \mathbb{C}$ and clearly, we obtain $q_0 \leq |\zeta_0|$. Because of the monotonicity of $\mathcal{J}_\nu(A)$ (i.e. $\mathcal{J}_{\nu+1}(A) \subseteq \mathcal{J}_\nu(A)$), ζ_{ρ_ν} eventually belong to $\mathcal{J}_\nu(A)$, $\forall\ \nu \in \mathbb{N}$, meaning that $\{\zeta_{\rho_\nu}\} \subseteq \bigcap_{\nu=1}^{\infty} \mathcal{J}_\nu(A) = \Lambda_k(A)$ and since $\Lambda_k(A)$ is closed, $\zeta_0 \in \Lambda_k(A)$. It implies $|\zeta_0| \leq r_k(A)$ and then $q_0 \leq r_k(A)$, a contradiction.

The second equality is apparent. □

Theorem 3.2. *Let $A \in \mathcal{M}_n$ and $\mathcal{J}_\nu(A) = \bigcap_{p=1}^{\nu} F(M_p^* A M_p)$, for some $M_p \in \mathcal{X}_{n-k+1}$. If $0 \notin \Lambda_k(A)$, then*

$$\widetilde{r}_k(A) = \lim_{\nu\to\infty} \inf\{|z| : z \in \mathcal{J}_\nu(A)\} = \sup_{\nu\in\mathbb{N}} \inf\{|z| : z \in \mathcal{J}_\nu(A)\}.$$

Proof. Obviously, $0 \notin \Lambda_k(A)$ indicates $\widetilde{r}_k(A) = \min\{|z| : z \in \Lambda_k(A)\}$ and by the relation (3.1), it is clear that

$$\|A\|_2 \geq r(A) \geq \widetilde{r}_k(A) = \min_{z\in\bigcap_{\nu=1}^{\infty}\mathcal{J}_\nu(A)} |z| \geq \inf_{z\in\mathcal{J}_\nu(A)} |z|.$$

Consequently, the sequence $t_\nu = \inf\{|z| : z \in \mathcal{J}_\nu(A)\}$, $\nu \in \mathbb{N}$, is nondecreasing and bounded and we have

$$\widetilde{r}_k(A) \geq \lim_{\nu\to\infty} t_\nu = t_0.$$

In a similar way as in Theorem 3.1, we will show that $\widetilde{r}_k(A) = \lim_{\nu\to\infty} t_\nu$. Suppose $\widetilde{r}_k(A) > t_0$, then $t_\nu \leq t_0 < \widetilde{r}_k(A) - \varepsilon$, for all $\nu \in \mathbb{N}$ and $\varepsilon > 0$. Considering the sequence $\{\widetilde{\zeta}_\nu\} \subseteq \mathcal{J}_\nu(A)$ such that $|\widetilde{\zeta}_\nu| \leq t_0$, let its subsequence $\{\widetilde{\zeta}_{s_\nu}\}$ converging to $\widetilde{\zeta}_0$, with $|\widetilde{\zeta}_0| \leq t_0$. Since $\{\mathcal{J}_\nu(A)\}$ is nonincreasing, $\widetilde{\zeta}_{s_\nu}$ eventually belong to $\mathcal{J}_\nu(A)$, $\forall\ \nu \in \mathbb{N}$, establishing $\{\widetilde{\zeta}_{s_\nu}\} \subseteq \bigcap_{\nu\in\mathbb{N}} \mathcal{J}_\nu(A) = \Lambda_k(A)$. Hence, we conclude $\widetilde{\zeta}_0 \in \bigcap_{\nu=1}^{\infty} \mathcal{J}_\nu(A) = \Lambda_k(A)$, i.e. $t_0 \geq |\widetilde{\zeta}_0| \geq \widetilde{r}_k(A)$, absurd.

The second equality is trivial. □

The next proposition asserts a lower and an upper bound for $r_k(A)$ and $\widetilde{r}_k(A)$, respectively.

Proposition 3.3. *Let $A \in \mathcal{M}_n$ and $M_p \in \mathcal{X}_{n-k+1}$, $p \in \mathbb{N}$, then*

$$r_k(A) \leq \inf_{p\in\mathbb{N}} r(M_p^* A M_p).$$

If $0 \notin \Lambda_k(A)$, then

$$\widetilde{r}_k(A) \geq \inf_{p\in\mathbb{N}} \widetilde{r}(M_p^* A M_p).$$

Proof. By Theorem 2.2, we obtain $\partial\Lambda_k(A) \subseteq \Lambda_k(A) \subseteq F(M_p^* A M_p)$ for all $p \in \mathbb{N}$. Then

$$r_k(A) = \max\{|z| : z \in \Lambda_k(A)\} \leq \max\{|z| : z \in F(M_p^* A M_p)\} = r(M_p^* A M_p).$$

Denoting by $c(M_p^* A M_p) = \min\{|z| : z \in F(M_p^* A M_p)\}$ for all $p \in \mathbb{N}$, we have

$$\widetilde{r}_k(A) \geq \min\{|z| : z \in \Lambda_k(A)\} \geq c(M_p^* A M_p).$$

Since $0 \leq c(M_p^* A M_p) \leq \widetilde{r}(M_p^* A M_p) \leq r(M_p^* A M_p) \leq \|A\|_2$ for any $p \in \mathbb{N}$, immediately, we obtain

$$r_k(A) \leq \inf_{p\in\mathbb{N}} r(M_p^* A M_p) \text{ and } \widetilde{r}_k(A) \geq \sup_{p\in\mathbb{N}} c(M_p^* A M_p).$$

If $0 \notin \Lambda_k(A)$, then by Theorem 2.2, $0 \notin F(M_l^* A M_l)$ for some $l \in \mathbb{N}$, $M_l \in \mathcal{X}_{n-k+1}$ and $c(M_l^* A M_l) = \widetilde{r}(M_l^* A M_l)$. Hence

$$\widetilde{r}_k(A) \geq \sup_{p\in\mathbb{N}} c(M_p^* A M_p) \geq \widetilde{r}(M_l^* A M_l) \geq \inf_{p\in\mathbb{N}} \widetilde{r}(M_p^* A M_p).$$

□

The numerical radius function $r(\cdot) : \mathcal{M}_n \to \mathbb{R}_+$ is not a matrix norm, nevertheless, it satisfies the power inequality $r(A^m) \leq [r(A)]^m$, for all positive integers m, which is utilized for stability issues of several iterative methods [2, 5]. On the other hand, the k-rank numerical radius fails to satisfy the power inequality, as the next counterexample reveals.

Example 3.4. Let the matrix $A = \begin{bmatrix} 1.8 & 2 & 3 & 4 \\ 0 & 0.8{+}\mathrm{i} & 0 & \mathrm{i} \\ -2 & 1 & -1.2 & 1 \\ 0 & 0 & 1 & 0.8 \end{bmatrix}$. Using Theorems 2.1 and 2.2, the set $\Lambda_2(A)$ is illustrated in the left part of Figure 1 by the uncovered area inside the figure. Clearly, it is included in the unit circular disc, which indicates that $r_2(A) < 1$. On the other hand, the set $\Lambda_2(A^2)$, illustrated in the right part of Figure 1 with the same manner, is not bounded by the unit circle and thus $r_2(A^2) > 1$. Obviously, $[r_2(A)]^2 < 1 < r_2(A^2)$.

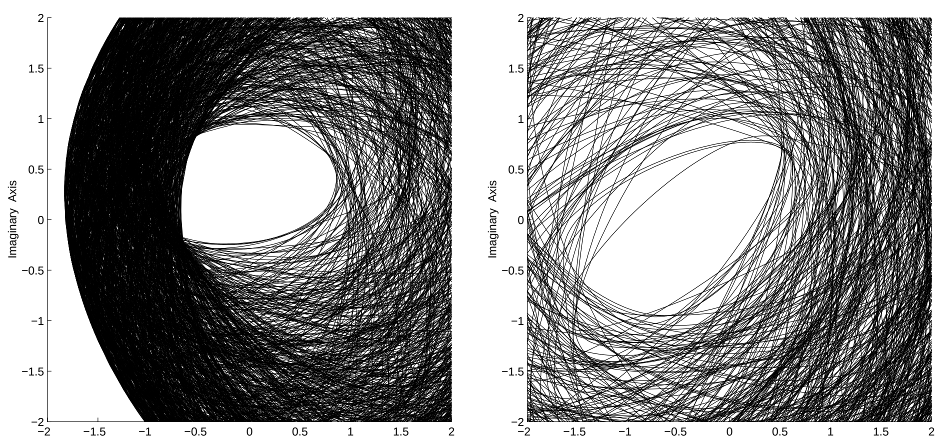

FIGURE 1. The "white" bounded areas inside the figures depict the sets $\Lambda_2(A)$ (left) and $\Lambda_2(A^2)$ (right).

The results developed in this paper draw attention to the rank-k numerical range $\Lambda_k(L(\lambda))$ of a matrix polynomial $L(\lambda) = \sum_{i=0}^{m} A_i \lambda^i$ ($A_i \in \mathcal{M}_n$), which has been extensively studied in [3, 4]. It is worth noting that Theorem 2.2 can be also generalized in the case of $L(\lambda)$, which follows readily from the proof. Hence, the rank-k numerical radii of $\Lambda_k(L(\lambda))$ can be elaborated with the same spirit as here [1].

Appendix A.

Following we provide *another construction* of a family of $n \times (n-k+1)$ isometries $\{M_\nu : \nu \in \mathbb{N}\}$ presented in Theorem 2.2.

Proof. By Theorem 2.1, we have

$$\Lambda_k(A) = \bigcap_{M \in \mathcal{X}_{n-k+1}} F(M^* A M), \tag{A.1}$$

which is known to be a compact and convex subset of $\mathbb{C}$. For any $n \times (n-k+1)$ isometry M_ν ($\nu \in \mathbb{N}$), we have $\Lambda_k(A) \subseteq F(M_\nu^* A M_\nu)$ for all $\nu \in \mathbb{N}$ and thus,

$$\Lambda_k(A) \subseteq \bigcap_{\nu \in \mathbb{N}} F(M_\nu^* A M_\nu). \tag{A.2}$$

In order to prove equality in the relation (A.2), we distinguish two cases for the interior of $\Lambda_k(A)$.

Suppose first that $\mathrm{int}\Lambda_k(A) \neq \emptyset$. Then by (A.2), we obtain

$$\emptyset \neq \mathrm{int}\Lambda_k(A) \subseteq \mathrm{int} \bigcap_{\nu \in \mathbb{N}} F(M_\nu^* A M_\nu)$$

and since $\bigcap_\nu F(M_\nu^* A M_\nu)$ is convex and closed, we establish

$$\overline{\mathrm{int} \bigcap_{\nu \in \mathbb{N}} F(M_\nu^* A M_\nu)} = \bigcap_{\nu \in \mathbb{N}} F(M_\nu^* A M_\nu), \tag{A.3}$$

where $\overline{\ \cdot\ }$ denotes the closure of a set. Thus, combining the relations (A.2) and (A.3), we have

$$\Lambda_k(A) \subseteq \overline{\mathrm{int} \bigcap_{\nu \in \mathbb{N}} F(M_\nu^* A M_\nu)}. \tag{A.4}$$

Further, we claim that $\mathrm{int} \bigcap_\nu F(M_\nu^* A M_\nu) \subseteq \Lambda_k(A)$. Assume on the contrary that $z_0 \in \mathrm{int} \bigcap_\nu F(M_\nu^* A M_\nu)$ but $z_0 \notin \Lambda_k(A)$, then there exists an open neighborhood $\mathcal{B}(z_0, \varepsilon)$, with $\varepsilon > 0$, such that

$$\mathcal{B}(z_0, \varepsilon) \subset \bigcap_{\nu \in \mathbb{N}} F(M_\nu^* A M_\nu) \ \text{ and } \ \mathcal{B}(z_0, \varepsilon) \cap \Lambda_k(A) = \emptyset.$$

Then, the set $[\Lambda_k(A)]^c = \mathbb{C} \backslash \Lambda_k(A)$ is separable, as an open subset of the separable space $\mathbb{C}$ and let $\mathcal{Z}$ be a countable dense subset of $[\Lambda_k(A)]^c$ [8]. Therefore, there exists a sequence $\{z_p : p \in \mathbb{N}\}$ in $\mathcal{Z}$ such that $\lim_{p \to \infty} z_p = z_0$ and $z_p \in \mathcal{B}(z_0, \varepsilon)$. Moreover, $z_p \in [\Lambda_k(A)]^c$ and by (A.1), it follows that for any p correspond indices $j_p \in \mathbb{N}$ such that $z_p \notin F(M_{j_p}^* A M_{j_p})$. Thus $z_p \notin \bigcap_{p \in \mathbb{N}} F(M_{j_p}^* A M_{j_p})$, which is absurd, since $z_p \in \mathcal{B}(z_0, \varepsilon) \subset \bigcap_{\nu \in \mathbb{N}} F(M_\nu^* A M_\nu)$. Hence $z_0 \in \Lambda_k(A)$, verifying our claim and we obtain

$$\overline{\mathrm{int} \bigcap_{\nu \in \mathbb{N}} F(M_\nu^* A M_\nu)} \subseteq \overline{\Lambda_k(A)} = \Lambda_k(A). \tag{A.5}$$

By (A.3), (A.4) and (A.5), the required equality is asserted.

Consider now that $\Lambda_k(A)$ has no interior points, namely, it is a line segment or a singleton. Then there is a suitable affine subspace $\mathcal{V}$ of $\mathbb{C}$ such that $\Lambda_k(A) \subseteq \mathcal{V}$ and with respect to the subspace topology, we have $\mathrm{int}\Lambda_k(A) \neq \emptyset$ and $\mathcal{V} \setminus \Lambda_k(A)$

be separable. Following the same arguments as above, let $\widetilde{\mathcal{Z}}$ be a countable dense subset of $\mathcal{V} \setminus \Lambda_k(A)$. Hence, there is a sequence $\{\widetilde{z}_q : q \in \mathbb{N}\}$ in $\widetilde{\mathcal{Z}}$ converging to z_0 and $\widetilde{z}_q \in \mathcal{B}(z_0, \varepsilon) \subset \bigcap_{\nu \in \mathbb{N}} F(M_\nu^* A M_\nu)$. On the other hand, by (A.1), we have $\widetilde{z}_q \notin \bigcap_{q \in \mathbb{N}} F(M_{i_q}^* A M_{i_q})$ for some indices $i_q \in \mathbb{N}$. Clearly, we are led to a contradiction and we deduce $\bigcap_{\nu \in \mathbb{N}} F(M_\nu^* A M_\nu) \subseteq \Lambda_k(A)$. Hence, with (A.2), we conclude

$$\Lambda_k(A) = \bigcap_{\nu \in \mathbb{N}} F(M_\nu^* A M_\nu).$$

□

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References

1. M. Adam, J. Maroulas and P. Psarrakos, *On the numerical range of rational matrix functions*, Linear and Multilinear Algebra **50** (2002), no. 1, 75–89.
2. T. Ando, *Structure of operators with numerical radius one*, Acta Scientia Mathematica (Szeged) **34** (1973, 11–15.
3. Aik. Aretaki, *Higher rank numerical ranges of nonnegative matrices and matrix polynomials*, Ph.D. Thesis, National Technical University of Athens, Greece, 2011.
4. Aik. Aretaki and J. Maroulas, *The higher rank numerical range of matrix polynomials*, 10th workshop on Numerical Ranges and Numerical Radii, Krakow, Poland, 2010, preprint http://arxiv.org/1104.1341v1 [math.RA], 2011, submitted for publication.
5. R.A. Horn and C.R. Johnson, *Topics in Matrix Analysis*, Cambridge University Press, Cambridge, 1991.
6. C.K. Li, Y.T. Poon and N.S. Sze, *Higher rank numerical ranges and low rank pertubations of quantum channels*, J. Mathematical Analysis and Applications, **348** (2008), 843–855.
7. C.K. Li and N.S. Sze, *Canonical forms, higher rank numerical ranges, totally isotropic subspaces, and matrix equations*, Proceedings of the American Mathematical Society, **136** (2008), 3013–3023.
8. J.R. Munkres, *Topology*, 2nd. Edition, Prentice Hall, 1975.
9. H.J. Woerdeman, *The higher rank numerical range is convex*, Linear and Multilinear Algebra, **56** (2007), no. 1, 65–67.

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